Triplex numbers and triplex roots of polynomials

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Abstract

In this paper, we define the triplex numbers, a hypercomplex number system with some similarities to the complex numbers. We construct an explicit isomorphism between $\mathbb{T}$ and $\mathbb{R} \times \mathbb{C}$, which enables one to study the properties of the triplex numbers in a natural setting.

In particular, we demonstrate a one-to-one correspondence between the sets of triplex roots of polynomial functions with coefficients in $\mathbb{R}$ and the sets of roots in $\mathbb{R} \times \mathbb{C}$. As one consequence of this correspondence, we find that given a polynomial $f(x)$ with coefficients in $\mathbb{R}$, the set of roots in the triplex numbers has cardinality $|X_0| \cdot |Z_0|$, where X_0 and Z_0 are the sets of real and complex roots, respectively.

1 Motivation

The *triplex numbers* are a hypercomplex number system obtained by adjoining to $\mathbb{R}$ two elements $i \notin \mathbb{R}$ and $j \notin \mathbb{R}$ which satisfy the following relations:

$$\begin{aligned} i^2 &= j \\ j^2 &= i \\ ij = ji &= 1. \end{aligned}$$

Arithmetic is carried out in the usual fashion, while keeping in mind to obey the above rules for dealing with the new units. An example calculation, where we multiply the triplex numbers $1 + j$ and $i + j$ together:

$$\begin{aligned} (1 + j)(i + j) &= i + j + ij + j^2 \\ &= i + j + 1 + i \\ &= 1 + 2i + j. \end{aligned}$$

Any calculation involving addition and multiplication can be carried out with triplex numbers, the result of which can always be "collapsed" to a triplex number of the form $a + bi + cj$. To use more precise terminology, the triplex numbers are closed under addition and multiplication. This can be seen clearly from Definition 2.7 and Definition 2.8.

It is also possible to perform division with triplex numbers. However, for reasons that will become clear later in this paper, dividing with triplex numbers comes with some caveats. It is something that must be done with the utmost care and precision, much like folding a shirt in the correct manner so as to prevent the formation of wrinkles.

Nevertheless, let us pause for a moment and reflect on what it is we have just seen. The triplex numbers are a system of numbers which extend the real numbers, much like the complex numbers. One could say that the complex numbers were borne from a sort of quest to give algebraic closure to the real numbers. Similarly, more advanced number systems such as the quaternions were invented to facilitate solving of problems that were difficult with real numbers and complex numbers alone.

This begs the question: Do the triplex numbers offer anything of value that the real numbers and the complex numbers do not? To give our inquiry some clearer direction, we will consider polynomials over $\mathbb{R}$, that is, polynomials $f(x) \in \mathbb{R}[x]$ of the form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where $a_0, ..., a_n \in \mathbb{R}$, and $n \in \mathbb{N}$ is the degree of the polynomial.

It is a well-known fact that $\mathbb{C}$ is an algebraically closed field, that is to say, polynomials with complex (and hence, also with real) coefficients always have at least one complex solution. In fact, counting multiplicity, a polynomial of degree n has exactly n complex roots, which is certainly a much more profound insight than the fact that there is always at least one. Nevertheless, it can be shown that these two conditions are in fact equivalent. This fact about polynomial roots is occasionally referred to as the "Fundamental theorem of algebra."

The implications of this fact for our investigation of the triplex numbers are multifold. For one, we would certainly be hard-pressed to find a polynomial equation not solvable in $\mathbb{C}$ which turns out to be solvable in $\mathbb{T}$, because such a polynomial cannot exist. However, it may certainly be the case that there exists some polynomial which is not solvable in $\mathbb{R}$, but which turns out to be solvable in $\mathbb{T}$. It may also be the case that certain kinds of polynomial equations become easier to solve due to the properties of $\mathbb{T}$, perhaps resulting in formulas which are simpler and easier to work with than their complex-valued counterparts.

For these reasons, we direct our attention to the question of triplex roots of polynomials. It may seem natural to start with polynomial equations of small degree, perhaps quadratic or cubic equations, and attempt to derive some kind of formula that can be used to find triplex solutions based on the rules for triplex addition and multiplication. This is certainly possible, but it would undoubtedly be a painstaking ordeal. Arithmetic with triplex numbers is, after all, at least $3/2$ times as painstaking as arithmetic with complex numbers.

Instead, one may consider whether more abstract ideas can be applied to help us understand the structure of the triplex numbers. Later in this paper, we will use sophisticated machinery to prove some general facts about triplex roots of polynomials. As it turns out, if a polynomial $f(x)$ has no real roots, then $f(x)$ has no roots in the triplex numbers, either. Perhaps disappointingly, this means that the triplex numbers are strictly inferior to the complex numbers, at least in terms of the quantity of polynomials which can be solved.

However, if $f(x)$ has real roots, or some mixture of real roots and complex roots, then there are actually numerous triplex roots. In fact, in this situation, the number of triplex roots is generally greater than both the number of real roots and the number of complex roots.

Let us consider the polynomial $f(x) = x^7 + x^6 - 6x^5 - x^2 - x + 6$. There are three real roots and four strictly complex roots. We will not list them explicitly here, but we assure the reader that they exist. It turns out that $f(x)$ has a grand total of 18 strictly triplex roots. So as to avoid cluttering the page, we again invoke the reader's trust as to the existence of all of these roots.

However, we invite the reader to check that

$$\begin{aligned}\zeta_1 &= -\frac{1}{3} + \frac{2}{3}i + \frac{2}{3}j \\ \zeta_2 &= \frac{1}{3} - \frac{5}{3}i - \frac{5}{3}j\end{aligned}$$

are indeed roots of $f(x)$.

Triplex roots of polynomials have another interesting aspect of note. One can define an operation analogous to complex conjugation on the triplex numbers, namely the action of weak conjugation. The weak conjugate of a triplex number is found by swapping the i-component and j-component of the number. It turns out that if a polynomial $f(x)$ has some triplex root, say ζ_1, then the weak conjugate of ζ_1 must also be a root of $f(x)$.

Applying this fact to the example considered above is unenlightening, because ζ_1 and ζ_2 are invariant under weak conjugation. In this paper, we refer to such triplex numbers as *flat.*

Consider instead the polynomial $g(x) = x^3 - 4x^2 + 6x - 24$. The reader may check that

$$\tau_1 = \frac{4}{3} + \left(\frac{4}{3} + \sqrt{2}\right) i + \left(\frac{4}{3} - \sqrt{2}\right) j$$

as well as the weak conjugate

$$(\tau_1)^- = \frac{4}{3} + \left(\frac{4}{3} - \sqrt{2}\right) i + \left(\frac{4}{3} + \sqrt{2}\right) j$$

are both roots of $g(x)$. This illustrates that triplex roots of polynomials always come in conjugate pairs. This is a property shared with the complex numbers, though the notion of conjugation differs with the triplex numbers.

Perhaps unsurprisingly, there is a close relationship between the triplex roots of a polynomial and the real and complex roots. It is our aim to elucidate this relationship. We do so chiefly in Section 4.

In the next few sections, we provide some needed definitions, describe preliminary notation, and explore some basic properties of $\mathbb{T}$ through the lens of a more intuitive number system.

2 Preliminaries

Definition 2.1. A *triplex number* is an ordered triplet of numbers, written

$$t := (a, b, c),$$

where $a, b, c \in \mathbb{R}$.

We will use both the ordered triplet notation, as well as its familiar equivalent notation

$$t = a + bi + cj$$

interchangeably without mention.

Definition 2.2. Given a triplex number $t = (a, b, c)$ with $a, b, c \in \mathbb{R}$, a is referred to as the real component, b is referred to as the i-component, and c is referred to as the j-component.

Definition 2.3. A triplex number of the form $t = (a, k, k)$ with $a, k \in \mathbb{R}$ is referred to as *flat.*

Definition 2.4. 1, i and j are referred to as the *basis elements of* $\mathbb{T}$. The set

$$\mathcal{B} := \{1, i, j\}$$

denotes the set of basis elements of $\mathbb{T}$.

Definition 2.5. The set of all triplex numbers is denoted

$$\mathbb{T} := \{(a, b, c) \mid a, b, c \in \mathbb{R}\}.$$

Definition 2.6. Two triplex numbers $t_1 = (a_1, b_1, c_1)$ and $t_2 = (a_2, b_2, c_2)$ are said to be *equal* if their corresponding real number entries are equal. That is,

$$t_1 = t_2 \iff a_1 = a_2 \text{ and } b_1 = b_2 \text{ and } c_1 = c_2.$$

It is straightforward to check that triplex equality is reflexive, symmetric, and transitive. Hence, triplex equality defines an equivalence relation on $\mathbb{T}$.

Definition 2.7. *Triplex addition* is defined as follows: given two triplex numbers $t_1 = (a_1, b_1, c_2)$ and $t_2 = (a_2, b_2, c_2)$, then the sum is given by the map $t_1 + t_2 : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ defined by

$$t_1 + t_2 \longmapsto (a_1 + a_2, b_1 + b_2, c_1 + c_2).$$

From Definition 2.7, it immediately follows that addition is associative and commutative. The additive identity is $(0, 0, 0)$:

$$(a, b, c) + (0, 0, 0) = (a, b, c).$$

Thus, $(\mathbb{T}, +)$ forms an abelian group. Moreover, as an additive group $(\mathbb{T}, +)$ is isomorphic to $\mathbb{R}^3$.

Definition 2.8. *Triplex multiplication* is defined as follows: given two triplex numbers $t_1 = (a_1, b_1, c_2)$ and $t_2 = (a_2, b_2, c_2)$, then the product $t_1 * t_2$ is given by the map $t_1 * t_2 : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ defined by

$$t_1 * t_2 \longmapsto (a_1a_2 + b_1c_2 + c_1b_2, a_1b_2 + b_1a_2 + c_1c_2, a_1c_2 + b_1b_2 + c_1a_2).$$

It follows from Definition 2.8 that triplex multiplication is associative, commutative, and distributes over addition. The multiplicative identity is $(1, 0, 0)$:

$$(a, b, c) * (1, 0, 0) = (a, b, c).$$

Thus, $(\mathbb{T}, +, *)$ forms a commutative ring with unity, hereafter denoted simply $\mathbb{T}$.

Definition 2.9. Scalar multiplication for triplex numbers is defined as follows: given a scalar $x \in \mathbb{R}$ and a triplex number $t = (a, b, c)$, the scalar multiple is given by

$$xt := (xa, xb, xc).$$

From Definition 2.9 it immediately follows that scalar multiplication is distributive with respect to both triplex addition and addition on $\mathbb{R}$. Hence, $\mathbb{T}$ forms a vector space over $\mathbb{R}$. As a vector space, $\mathbb{T}$ is isomorphic to $\mathbb{R}^3$.

Together with the bilinear product defined in Definition 2.8, $\mathbb{T}$ forms a unital associative algebra over the reals.

3 Properties of the triplex numbers

3.1 Relationship with real-complex numbers

There are several clues that there are may be alternative ways to view the triplex numbers. Consider the polynomial $f(x) = x^3 - 1$. Its three complex roots are the cubic roots of unity:

$$\begin{aligned} x_1 &= 1 \\ x_2 &= e^{\frac{2}{3}\pi i} \\ x_3 &= e^{\frac{4}{3}\pi i}. \end{aligned}$$

One may observe that these roots under multiplication exhibit the same behavior as the basis elements of $\mathbb{T}$ under multiplication:

$$\begin{aligned} x_2^2 &= x_3 \\ x_3^2 &= x_2 \\ x_2 x_3 = x_3 x_2 &= 1. \end{aligned}$$

A classic result of elementary ring theory is the fact that $\mathbb{C} \cong \mathbb{R}[x]/\left\langle x^2+1\right\rangle$. In a sense, one may view this as a construction of the complex numbers from the reals by "identifying" the square of the indeterminate with -1 in the polynomial ring. Elements of $\mathbb{R}[x]/\left\langle x^2+1\right\rangle$ have the form $bx+a+\left\langle x^2+1\right\rangle$, where x behaves identically to the imaginary unit i. The quotient ring turns out to be a field which is isomorphic to the complex numbers.

It stands to reason that a similar construction with the polynomial x^3-1 ought to be possible. The quotient ring $\mathbb{R}[x]/\left\langle x^3-1\right\rangle$ should be a ring in which one has adjoined to $\mathbb{R}$ two new elements which behave exactly as the complex roots of x^3-1 do. The resulting quotient ring should then be functionally the same as $\mathbb{T}$.

We will verify this relationship formally in Proposition 3.1, but before we do so, let us have one more insight. Equipped with ordinary multiplication, the set $G := \{x_1, x_2, x_3\}$ of cubic roots of unity forms a group, namely the cyclic group of order 3. This is illustrated in Figure 3.1.

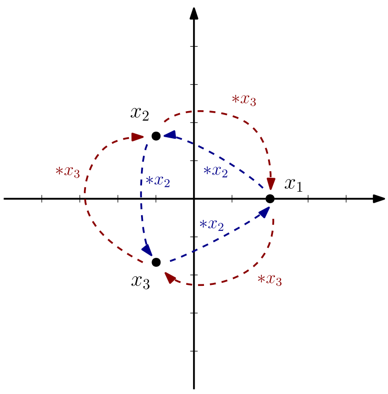

Figure 3.1: The group G acting on itself by multiplication, with group actions by x_2 and x_3 shown in blue and red, respectively.

Curiously, in our description of the quotient ring $\mathbb{R}[x]/\left\langle x^3-1\right\rangle$, we have simultaneously described $\mathbb{R}[C_3]$, the group ring of C_3 over $\mathbb{R}$. As it turns out, they are isomorphic. Proposition 5.1 states far more generally that, given an arbitrary field K, there is an isomorphism of K-algebras $K[x]/\left\langle x^n-1\right\rangle \cong K[C_n]$. This is fairly easy to visualize: When the roots of a polynomial form a group under multiplication, the quotient ring becomes the group ring of the corresponding group over K. If we take K to be the field of complex numbers, then the only multiplicative subgroups are the cyclic subgroups of order n, corresponding precisely so the sets of nth roots of unity. In this case, the polynomial must take the form $ax^n - a$, where $a \in \mathbb{C}$.

Let us move our attention back to the triplex numbers and verify that $\mathbb{T}$ and $\mathbb{R}[x]/\left\langle x^3-1\right\rangle$ are in fact isomorphic. For the sake of brevity, $[\cdot]$ is used to denote equivalence class when there is no real risk of confusion.

Proposition 3.1. *The map* $\phi : \mathbb{T} \to \mathbb{R}[x]/\left\langle x^3 - 1\right\rangle$ *defined by*

$$\phi : u + vi + wj \longmapsto \left[u + vx + wx^2\right]$$

where $u, v, w \in \mathbb{R}$, *is an isomorphism of* $\mathbb{R}$*-algebras.*

Proof. Trivially, ϕ preserves the unity of both rings. It is straightforward to check that ϕ is surjective, injective and therefore a bijection. It is similarly straightforward (though somewhat laborious) to check that ϕ respects the addition and multiplication operations of both rings. Clearly, ϕ respects scalar multiplication as well. We will omit the laborious computations here. □

Proposition 3.1 allows us to continue our study of the structure and properties of $\mathbb{T}$ through the quotient ring $\mathbb{R}[x]/\left\langle x^3 - 1\right\rangle$. Our next step is to note that the polynomial $x^3 - 1$ is reducible over $\mathbb{R}$:

$$x^3 - 1 = (x-1)(x^2 + x + 1).$$

The Chinese Remainder Theorem for rings then implies the following isomorphism:

$$\mathbb{R}[x]/\left\langle x^3 - 1\right\rangle \cong \mathbb{R}[x]/\left\langle x - 1\right\rangle \times \mathbb{R}[x]/\left\langle x^2 + x + 1\right\rangle.$$

We will construct the isomorphism explicitly soon, but for now, let us note that its existence gives us several properties of $\mathbb{T}$ "for free," without the need to conduct any triplex arithmetic whatsoever. In particular, we may conclude right away that $\mathbb{T}$ is not an integral domain, and hence not a field.

Corollary 3.1. $\mathbb{T}$ *is not an integral domain.*

Proof. The property of being an integral domain is preserved by isomorphisms, and the product ring $\mathbb{R}[x]/\left\langle x - 1\right\rangle \times \mathbb{R}[x]/\left\langle x^2 + x + 1\right\rangle$ is not an integral domain. As a fairly trivial example, consider the pairs[1] $(0, 1)$, $(1, 0)$ and their product:

$$(0, 1) * (1, 0) = (0, 0).$$

We see that $(0, 1)$ and $(1, 0)$ are zero divisors in the product ring, hence $\mathbb{T}$ is not an integral domain. □

As a result of this observation, some of the "nice" properties of working with our usual number systems $\mathbb{R}$ and $\mathbb{C}$ fail in the triplex numbers. This would include the cancellation property: for $a, b, c \in \mathbb{T}$, if we have $a \neq 0$, then $ab = ac$ does not imply $b = c$, as in the reals. In conclusion, we must be careful when "doing algebra" on the triplex numbers.

We continue now along our journey. For the sake of brevity, we will hereafter denote

$$\begin{aligned} Q &= \mathbb{R}[x]/\left\langle x^3 - 1\right\rangle \\ P_1 &= \mathbb{R}[x]/\left\langle x - 1\right\rangle \\ P_2 &= \mathbb{R}[x]/\left\langle x^2 + x + 1\right\rangle. \end{aligned}$$

Define the maps

$$\begin{aligned} \sigma_1 &: Q \longrightarrow P_1 \\ \sigma_2 &: Q \longrightarrow P_2 \end{aligned}$$

as follows:

$$\begin{aligned} \sigma_1 &: \left[u + vx + wx^2\right] \longmapsto [u + v + w] \\ \sigma_2 &: \left[u + vx + wx^2\right] \longmapsto [(u - w) + (v - w)x] \end{aligned}$$

where $[\cdot]$ denotes equivalence class in the respective quotient ring.

It is straightforward to check that σ_1 and σ_2 are $\mathbb{R}$-algebra homomorphisms. Note that $\ker \sigma_1$ is the set of all elements whose coefficients sum to 0, while $\ker \sigma_2$ is the set of all elements whose coefficients are equal.

Again, we define a set of maps

[1] To avoid unnecessary visual clutter, we have opted not to indicate $+\left\langle x - 1\right\rangle$ and $+\left\langle x^2 + x + 1\right\rangle$ everywhere these would normally appear. The reader is nevertheless reminded that each component in the pair is an element of the respective quotient ring.

$$
\begin{aligned}
\omega_1 &: P_1 \longrightarrow \mathbb{R} \\
\omega_2 &: P_2 \longrightarrow \mathbb{C}
\end{aligned}
$$

as follows:

$$
\begin{aligned}
\omega_1 &: [a] \longrightarrow a \\
\omega_2 &: [a+bx] \longrightarrow a - \frac{b}{2} + \frac{bi\sqrt{3}}{2}.
\end{aligned}
$$

One may check that ω_1 and ω_2 are $\mathbb{R}$-algebra isomorphisms. Again, we omit the laborious computations.

At this point, we take the product of maps $\langle \omega_1 \circ \sigma_1,\, \omega_2 \circ \sigma_2 \rangle$, obtaining a map from $\mathbb{R}[x]/\left\langle x^3-1 \right\rangle$ to $\mathbb{R} \times \mathbb{C}$.

Theorem 3.1. *Define the functions*

$$
\begin{aligned}
\alpha(u,v,w) &= u+v+w \\
\beta(u,v,w) &= u - \frac{v}{2} - \frac{w}{2} \\
\gamma(u,v,w) &= \frac{\sqrt{3}}{2}(v-w)
\end{aligned}
$$

where $u,v,w \in \mathbb{R}$. *Then the map* $\Psi : \mathbb{R}[x]/\left\langle x^3-1 \right\rangle \to \mathbb{R} \times \mathbb{C}$ *defined by*

$$\Psi : \left[u+vx+wx^2\right] \longmapsto (\alpha,\, \beta + i\gamma)$$

is an isomorphism of $\mathbb{R}$*-algebras.*

Proof. The map $\Psi = \langle \omega_1 \circ \sigma_1,\, \omega_2 \circ \sigma_2 \rangle$ has been constructed in such a fashion that the diagram in Figure 3.2 commutes. By the universal propery of the product, Ψ must be the unique such morphism. We have isomorphisms ω_1 and ω_2 between the factor algebras, hence by Proposition 5.2, the products must be isomorphic. It follows that Ψ must have an inverse which is also an $\mathbb{R}$-algebra homomorphism. $\square$

$$
\begin{tikzcd}
 & P_2 \arrow[rr, "\omega_2"] & & \mathbb{C} \\
\mathbb{T} \arrow[r, "\phi"] & Q \arrow[u, "\sigma_2"] \arrow[d, "\sigma_1"'] \arrow[rr, dashed, "\Psi"] & & \mathbb{R} \times \mathbb{C} \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] \\
 & P_1 \arrow[rr, "\omega_1"] & & \mathbb{R}
\end{tikzcd}
$$

Figure 3.2: Diagram showing maps from $\mathbb{T}$ to $\mathbb{R} \times \mathbb{C}$.

Theorem 3.2. *Define the functions*

$$
\begin{aligned}
u(\alpha,\beta,\gamma) &= \frac{1}{3}(\alpha + 2\beta) \\
v(\alpha,\beta,\gamma) &= \frac{1}{3}(\alpha - \beta + \sqrt{3}\gamma) \\
w(\alpha,\beta,\gamma) &= \frac{1}{3}(\alpha - \beta - \sqrt{3}\gamma)
\end{aligned}
$$

where $\alpha,\beta,\gamma \in \mathbb{R}$. *Then the map* $\Psi^{-1} : \mathbb{R} \times \mathbb{C} \to \mathbb{R}[x]/\left\langle x^3-1 \right\rangle$ *defined by*

$$\Psi^{-1} : (\alpha, \beta + i\gamma) \longmapsto \left[u(\alpha,\beta,\gamma) + v(\alpha,\beta,\gamma)x + w(\alpha,\beta,\gamma)x^2\right]$$

is an isomorphism of $\mathbb{R}$*-algebras. Moreover,* Ψ^{-1} *is the inverse of* Ψ.

Proof. Again we have constructed a map in such a fashion that the diagram in Figure 3.2, this time seen with the arrows turned around, commutes. We invoke Proposition 5.2, concluding that Ψ^{-1} is the inverse of Ψ. $\square$

For the rest of this paper, we define the map $\Phi : \mathbb{T} \to \mathbb{R} \times \mathbb{C}$ to be the composition $\Phi = \Psi \circ \phi$, where ϕ is as defined in the previous section, and $\Phi^{-1} : \mathbb{R} \times \mathbb{C} \to \mathbb{T}$ to be the composition $\Phi^{-1} = \phi^{-1} \circ \Psi^{-1}$. Clearly, Φ and Φ^{-1} are isomorphisms of $\mathbb{R}$-algebras.

The maps Φ and Φ^{-1} are quite useful. They provide a convenient means of converting back and forth between triplex numbers and elements of $\mathbb{R} \times \mathbb{C}$. These are numbers of the form $(\alpha, \beta + i\gamma)$, where $\alpha, \beta, \gamma \in \mathbb{R}$, hereafter referred to as real-complex numbers. It is often easier to ask and answer questions about the properties of $\mathbb{R} \times \mathbb{C}$, which is a ring with a much more intuitive arithmetic. Structure and properties which are preserved by isomorphisms will automatically be translatable to $\mathbb{T}$.

Φ and Φ^{-1} also show a clear relationship between the triplex conjugate and the complex conjugate of the number $\beta + i\gamma$ in the corresponding real-complex pair: given a triplex number, if v and w swap places, then the sign of γ is negated in the real-complex pair obtained by Φ; conversely, given a real-complex pair, if γ is negated, then v and w swap places in the triplex number obtained by Ψ^{-1}.

Any real-complex pair with $\gamma = 0$ corresponds to a triplex number with $v = w$. Hence, the set of triplex numbers which are invariant under the action of swapping v and w are those whose i-component and j-component are equal, and the set of corresponding complex numbers with $\gamma = 0$ are likewise invariant under this same action.

3.2 Zero divisors

By Corollary 3.1, $\mathbb{T}$ is not an integral domain, which indicates the presence of zero divisors. One may wonder as to what these zero divisors are, so that one may take the necessary precautions when "doing algebra."

If one simply plays around with triplex numbers for any length of time, one may stumble on some zero divisors. Consider the triplex numbers $\zeta_1 = 2 + 2i - 4j$ and $\zeta_2 = 1 + i + j$. We invite the reader to check that their product $(2 + 2i - 4j)(1 + i + j)$ is in fact zero. Hence, ζ_1 and ζ_2 are zero divisors.

The astute reader may have noticed that the sum of ζ_1's components is zero, while ζ_2 has components which are all equal. Carrying out the multiplication, we see that each of the respective components vanish. This would suggest at least two sets of zero divisors: triplex numbers whose components sum to 0, and triplex numbers whose components are all equal. Multiplying a triplex number from the first set by a triplex number from the second set should result in zero.

Indeed, one may select some other triplex numbers matching these criteria and verify that this seems to be the case. It remains to be shown however that these are the only zero divisors in $\mathbb{T}$. Do there exist other, less obvious zero divisors?

Our means of translating between triplex numbers and real-complex numbers permits a swift answer to this question. Observe that ζ_1 corresponds to the real-complex pair $(0,\ 1 + i\sqrt{3})$, while ζ_2 corresponds to $(3,\ 0)$. It is easy to see that in general, zero divisors in $\mathbb{R} \times \mathbb{C}$ must be of the form $(\alpha, 0)$ or $(0, \beta + i\gamma)$, where $\alpha, \beta, \gamma \in \mathbb{R}$. Translating to $\mathbb{T}$ via Φ^{-1}, we find that the zero divisors are precisely the sets described in the preceding paragraphs. Note that these sets are precisely $\ker \sigma_1 \circ \phi$ and $\ker \sigma_2 \circ \phi$, respectively, and that this relationship is actually rather unsurprising given the nature of the product ring.

3.3 Idempotents

In this subsection, we wish to find the idempotent elements of $\mathbb{T}$. Any unital ring has at least two idempotents, namely the additive and multiplicative identities. Hence, 0 and 1 are idempotent in $\mathbb{T}$.

To find the remaining idempotents, we will search for idempotent real-complex pairs, and then translate these numbers to triplex form by the map Φ^{-1}. Note that an element $t = u + vi + wj$ is idempotent if and only if $\alpha^2 = \alpha$ and $(\beta + i\gamma)^2 = \beta + i\gamma$ both hold for the corresponding real-complex pair.

It is known that the only idempotents in $\mathbb{R}$ are 0 and 1. Clearly, $\alpha \in \{0, 1\}$ are the only possibilities for the real part. The complex part reduces to solving the equation

$$\beta^2 - \beta - \gamma^2 + (2\beta\gamma - \gamma)i = 0$$

in β, γ. The equation is true if and only if $\beta^2 - \beta - \gamma^2 = 0$ and $2\beta\gamma - \gamma = 0$. The second condition implies $\beta = \frac{1}{2}$ or $\gamma = 0$, while the first condition implies $\gamma = \pm\sqrt{\beta^2 - \beta}$. Assuming that $\beta = \frac{1}{2}$, by the first condition γ is a complex number, a contradiction, so $\beta = \frac{1}{2}$ cannot be a solution. Hence, $\gamma = 0$, which then implies $\beta^2 = \beta$. It follows that $\beta = 0$ or $\beta = 1$. Hence, the only idempotents of $\mathbb{R} \times \mathbb{C}$ are the four elements $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$.

Each idempotent real-complex pair corresponds to an idempotent triplex number. Using Φ^{-1} to translate back to $\mathbb{T}$, we see that the idempotents are

$$\begin{aligned}
\eta_0 &= 0 \\
\eta_1 &= \frac{2}{3} - \frac{1}{3}i - \frac{1}{3}j \\
\eta_2 &= \frac{1}{3} + \frac{1}{3}i + \frac{1}{3}j \\
\eta_3 &= 1.
\end{aligned}$$

Note that η_1 is in $\ker \sigma_1 \circ \phi$, and η_2 is in $\ker \sigma_2 \circ \phi$.

3.4 Triplex conjugation

An essential feature of the regular complex numbers is the complex conjugate: given any complex number $z = x + iy$, the conjugate is defined to be $\bar{z} = x - iy$, and has a number of interesting properties. In particular, the complex norm may be defined in terms of the conjugate. The complex conjugate is among the first objects studied in complex analysis, and in our study of the triplex numbers it would seem a natural idea to define triplex conjugation in some meaningful way.

3.4.1 Weak conjugation

Definition 3.1. *Weak conjugation* is the map $\mathbb{T} \to \mathbb{T}$ defined by

$$(u + vi + wj)^- \longmapsto u + wi + vj.$$

We will variously denote the weak conjugate of a triplex number z by either z^- or $\mathrm{Wk}(z)$.

It is easy to check that weak conjugation is an automorphism of $\mathbb{T}$. In fact, by Proposition 3.3 it is the only nontrivial automorphism of $\mathbb{T}$. Moreover, weak conjugation fixes elements of $\mathbb{R}$.

Additionally, weak conjugation is an involution: for any triplex number $z = u + vi + wj$, we have that $(z^-)^- = z$. These are properties that weak conjugation shares with the regular complex conjugate. Note also that the sets $\ker \sigma_1$ and $\ker \sigma_2$ are closed under weak conjugation.

Weak conjugation with triplex numbers is considerably less useful than regular conjugation with complex numbers. For one, there is no immediately obvious way to define a norm using it. Nevertheless, weak conjugation has some interesting properties which make it valuable to study.

One can see that flat triplex numbers are invariant under weak conjugation:

$$(a + ki + kj)^- \longmapsto a + ki + kj.$$

With the complex numbers, the product of a complex number with its conjugate is strictly real. However, with the triplex numbers, we note that the product of a triplex number with its weak conjugate produces another triplex number. For a triplex number $z = u + vi + wj$, we have

$$zz^- = u^2 + v^2 + w^2 + (uv + vw + uw)i + (uv + vw + uw)j.$$

However, this product is always a flat triplex number, hence invariant under the action of weak conjugation. In other words, the following identity holds:

$$(zz^-)^- = zz^-$$

This property is shared with the complex conjugate.

Proposition 3.2. *The cube roots of* 1 *in* $\mathbb{T}$ *are the basis elements* $1, i, j$.

Proof. It is easy to see that the basis elements are cube roots of 1 in $\mathbb{T}$, but it remains to be shown that they are the only cube roots of 1 in $\mathbb{T}$.

A real-complex pair $(\alpha, \beta + i\gamma)$ where $\alpha, \beta, \gamma \in \mathbb{R}$ satisfies $(\alpha, \beta + i\gamma)^3 = 1$ if and only if

$$\begin{aligned}
\alpha^3 &= 1 \\
(\beta + i\gamma)^3 &= 1.
\end{aligned}$$

Hence, since α is real, $\alpha = 1$, and $\beta + i\gamma$ is a complex cube root of 1. So, the real-complex cube roots of 1 are

$$(1, 1)$$
$$(1, e^{\frac{2\pi i}{3}})$$
$$(1, e^{\frac{4\pi i}{3}}).$$

Using Φ^{-1} to translate back to $\mathbb{T}$, we see that these numbers correspond to the basis elements $1, i, j$ as desired. □

Proposition 3.3. *Weak conjugation is an automorphism of* $\mathbb{T}$. *Moreover, weak conjugation is the only nontrivial automorphism of* $\mathbb{T}$.

Proof. Let $\theta : \mathbb{T} \to \mathbb{T}$ be an automorphism of $\mathbb{T}$. Let $u + vi + wj$ be a triplex number, with $u, v, w \in \mathbb{R}$. Since $\mathbb{R}$ is a subring of $\mathbb{T}$, the restriction $\theta|_{\mathbb{R}}$ is an automorphism of $\mathbb{R}$. The only automorphism of $\mathbb{R}$ is the identity map, hence θ fixes $\mathbb{R}$. Then

$$\begin{aligned} \theta(u + vi + wj) &= \theta(u) + \theta(vi) + \theta(wj) \\ &= \theta(u) + \theta(v)\theta(i) + \theta(w)\theta(j) \\ &= u + v\theta(i) + w\theta(j). \end{aligned}$$

Hence, θ is uniquely determined by its action on the basis elements of $\mathbb{T}$.
Moreover, since θ must preserve 1, we see that

$$\begin{aligned} \theta(1) &= \theta(i^3) \\ 1 &= (\theta(i))^3. \end{aligned}$$

In other words, θ must send the basis elements i and j to cube roots of 1. By Proposition 3.2, the only cube roots of 1 are the basis elements themselves. Hence, the only automorphisms of $\mathbb{T}$ are the trivial automorphism sending every element to itself, and the weak conjugation map, which exchanges the basis elements i and j. □

3.4.2 Strong conjugation

Definition 3.2. *Strong conjugation* is the map $\mathbb{T} \to \mathbb{T}$ defined by

$$(u + vi + wj)^* \longmapsto u^2 - vw + (w^2 - uv)i + (v^2 - uw)j.$$

We will also denote strong conjugation in the same manner as the regular complex conjugate, with the use of the overbar: for a triplex number z, we denote the strong conjugate as either z^* or $\bar{z}$.
Strong conjugation takes η_1 to η_3, and η_2 to η_0, and η_0 to η_0, and η_3 to η_3. More generally, strong conjugation takes elements of $\ker \sigma_1$ to $\ker \sigma_2$, and elements of $\ker \sigma_2$ to 0.
It is easy to see that strong conjugation is not an automorphism of $\mathbb{T}$; this is a simple consequence of Proposition 3.3. It is also easy to see that in general, strong conjugation does not fix $\mathbb{R}$. In fact, strong conjugation restricted to the reals is simply the map taking each element to its square. Hence, strong conjugation is not an involution.
Strong conjugation does have one desirable property. Namely, the product of a triplex number with its strong conjugate is strictly real:

$$(u + vi + wj)(u + vi + wj)^* = u^3 + v^3 + w^3 - 3uvw.$$

This expression may be factored as

$$(u + v + w)(u^2 - vw + v^2 - uw + w^2 - uv).$$

As a result of this, the product of a triplex number $z = u + iv + jw$ with its strong conjugate is zero precisely when the sum of its components is zero, or when $u = v = w$.

Corollary 3.2. *The relationship between the weak and strong conjugates is characterized by the following identity:*

$$\mathrm{Wk}(\bar{z}) = \overline{\mathrm{Wk}(z)}.$$

Proof. The proof is left as an exercise to the reader. □

4 Triplex roots of polynomials

At this point, we turn our attention back to the question which captivated our interest in Section 1: how might one go about finding triplex roots of polynomial equations? Is there some kind of relationship between triplex roots of polynomials and the real and complex roots?

Indeed, the maps Φ and Φ^{-1} provide us with not only a means of easily generating triplex solutions to polynomial equations should their real and complex solutions already be known, but also allow us to concretize the relationship between solution sets of polynomial equations in $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{T}$.

Let us first describe some preliminary notation. For the remainder of this section, fix an arbitrary ring R with unity.

Definition 4.1. Let A be an algebra over R. Let

$$f(x) = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 + ... + a_n \cdot x^n$$

be a polynomial with coefficients $a_0, ..., a_n \in R$ and indeterminate x. An element $\zeta \in A$ is said to be a *root of* $f(x)$ *in* A if and only if

$$a_0 \cdot 1_A + a_1 \cdot \zeta + a_2 \cdot \zeta^2 + ... + a_n \cdot \zeta^n = 0_A$$

where 0_A and 1_A denote the additive and multiplicative identities of A, respectively.

Definition 4.2. The set of roots of a polynomial $f(x)$ in an R-algebra A is referred to as the *omega-set of* f *in* A, written $\Omega_f(A)$. More precisely:

$$\Omega_f(A) := \{x \in A \mid f(x) = 0_A\}.$$

Proposition 4.1. *Let A and B be R-algebras and the map $\phi : A \to B$ a homomorphism of R-algebras. If $\zeta \in A$ is a root of $f(x)$ in A then $\phi(\zeta)$ is necessarily a root of $f(x)$ in B.*

Proof. $\zeta \in A$ is a root of $f(x)$ implies that

$$\phi(a_0 \cdot 1_A + a_1 \cdot \zeta + a_2 \cdot \zeta^2 + ... + a_n \cdot \zeta^n) = \phi(0_A).$$

Because ϕ is an R-algebra homomorphism, ϕ commutes with addition, multiplication, and scalar multiplication. Moreover, for any $k \in \mathbb{N}$, $\phi(\zeta^k) = \phi(\zeta)^k$. Hence, the above condition is equivalent to

$$a_0 \cdot 1_B + a_1 \cdot \phi(\zeta) + a_2 \cdot \phi(\zeta)^2 + ... + a_n \cdot \phi(\zeta)^n = 0_B$$

which means that $\phi(\zeta)$ is a root of $f(x)$ in B. $\square$

As a consequence of Proposition 4.1, an R-algebra homomorphism $\phi : A \to B$ induces a mapping $\eta_f(\phi) : \Omega_f(A) \to \Omega_f(B)$, which is simply the restriction of ϕ to $\Omega_f(A)$. If ϕ is injective, then this mapping is necessarily injective, and if ϕ is bijective, then this mapping is necessarily bijective.

Lemma 4.1. *If $t \in \mathbb{T}$ is a root of $f(x)$ in $\mathbb{T}$ then there exists some $z_0 \in \mathbb{C}$ and $x_0 \in \mathbb{R}$ which are roots of $f(x)$ in $\mathbb{C}$ and $\mathbb{R}$, respectively. Conversely, let $z_0 \in \mathbb{C}$ and $x_0 \in \mathbb{R}$ be roots of $f(x)$ in $\mathbb{C}$ and $\mathbb{R}$, respectively. Then $\Phi^{-1}[(x_0, z_0)]$ is a root of $f(x)$ in $\mathbb{T}$.*

Proof. Let $t \in \mathbb{T}$ be a root of $f(x)$ in $\mathbb{T}$. By Proposition 4.1, this corresponds to precisely one real-complex root, namely $\Phi(t) = (x_0, z_0)$, where $x_0 \in \mathbb{R}$ and $z_0 \in \mathbb{C}$. Because we have morphisms from $\mathbb{T}$ onto $\mathbb{R}$ and $\mathbb{C}$, namely the projection maps after mapping to $\mathbb{R} \times \mathbb{C}$, Proposition 4.1 implies that x_0 and z_0 are roots of $f(x)$ in $\mathbb{R}$ and $\mathbb{C}$, respectively.

To convince yourself of the truth of the converse direction, one may begin with $x_0 \in \mathbb{R}$ and $z_0 \in \mathbb{C}$ which are roots of $f(x)$ in $\mathbb{R}$ and $\mathbb{C}$, respectively. Clearly, (x_0, z_0) is a root of $f(x)$ in $\mathbb{R} \times \mathbb{C}$. By Proposition 4.1, $\Phi^{-1}[(x_0, z_0)]$ is a root of $f(x)$ in $\mathbb{T}$. $\square$

Lemma 4.2. *Let X_0 be the set of roots of $f(x)$ in $\mathbb{R}$ and Z_0 the set of roots of $f(x)$ in $\mathbb{C}$ (so that X_0 is a subset of Z_0). Then T_0, the set of roots of $f(x)$ in $\mathbb{T}$, has cardinality $|X_0| \cdot |Z_0|$.*

Proof. The proof follows more or less immediately from Lemma 4.1. Counting the roots of $f(x)$ in $\mathbb{T}$ amounts to counting the roots in $\mathbb{R} \times \mathbb{C}$, which amounts to counting the possible ways that one can arrange (x_0, z_0) with $x_0 \in \mathbb{R}$ and $z_0 \in \mathbb{C}$ roots of $f(x)$ in $\mathbb{R}$ and $\mathbb{C}$, respectively. $\square$

Lemma 4.3. *If $f(x)$ has no roots in $\mathbb{R}$, then $f(x)$ has no roots in $\mathbb{T}$.*

Proof. This is an obvious consequence of Lemma 4.2. $\square$

Proposition 4.2. *If $t = u + vi + wj \in \mathbb{T}$ is a root of $f(x)$, then t^- is also a root of $f(x)$.*

Proof. Weak conjugation is an automorphism of $\mathbb{T}$, hence by Proposition 4.1, t^- is a root of $f(x)$ in $\mathbb{T}$. $\square$

Proposition 4.2 shows us that triplex solutions to polynomial equations always come in conjugate pairs, which is a property shared with complex numbers.

5 Miscellaneous results

In this section, we have collected some miscellaneous results which may have been used to justify side notes or motivate certain proof steps. Proposition 5.1 for example is a vast generalization of the idea that the triplex numbers may be viewed simultaneously as the quotient ring $\mathbb{R}[x]/\left\langle x^3 - 1\right\rangle$ and the group ring of C_3 over $\mathbb{R}$.

Proposition 5.1. *Let K be a field, and $K[x]$ the ring of polynomials over K with indeterminate x. For any $n \in \mathbb{N}$ there is an isomorphism of K-algebras*

$$K[x]/\left\langle x^n - 1\right\rangle \cong K[C_n]$$

where C_n denotes the cyclic group of order n, and $K[C_n]$ is the group ring of C_n over K.

Proof. Denote the elements of C_n as

$$C_n = \{g_0,\, g_1,\, g_2,\, \dots,\, g_{n-1}\},$$

where g_0 denotes the identity, and g_1 the generator.

Define a map $\iota : C_n \to K[x]/\left\langle x^n - 1\right\rangle^{\times}$ as follows:

$$\iota : g_k \longmapsto x^k + \left\langle x^n - 1\right\rangle.$$

Because $x^{n-k}x^k = x^n = 1 + \langle x^n - 1\rangle$, $x^k + \langle x^n - 1\rangle$ is a unit. It is easy to check that ι is a homomorphism of groups. Moreover, ι is injective.

Let S be some arbitrary K-algebra and $f : C_n \to S^{\times}$ a K-algebra homomorphism. Define a map $\bar{f} : K[x]/\langle x^n - 1\rangle \to S$ as follows:

$$\bar{f} : p(x) + \left\langle x^n - 1\right\rangle \longmapsto p(f(g_1)).$$

Intuitively, $\bar{f}$ evaluates a polynomial $p(x)$, which itself is a K-linear combination of powers of x, at the image of the generator g_1 under f. The result of this evaluation is a K-linear combination of powers of this image, hence "extending" the map f to S.

We must show that $\bar{f}$ is well-defined. Let $p(x), q(x) + \langle x^n - 1\rangle \in K[x]/\langle x^n - 1\rangle$ such that

$$p(x) + \left\langle x^n - 1\right\rangle = q(x) + \left\langle x^n - 1\right\rangle.$$

This is equivalent to the condition that $p(x) - q(x) = r(x)(x^n - 1)$ for some $r(x) \in K[x]$. Then

$$\bar{f}(p(x) + \left\langle x^n - 1\right\rangle) = p(f(g_1))$$
$$\bar{f}(q(x) + \left\langle x^n - 1\right\rangle) = q(f(g_1))$$

and

$$p(f(g_1)) - q(f(g_1)) = r(f(g_1))((f(g_1))^n - 1).$$

Using the fact that f is a K-algebra homomorphism, we can simplify the condition to

$$p(f(g_1)) = q(f(g_1)),$$

showing that the images of $p(x)$ and $q(x)$ under $\bar{f}$ are equal. Hence $\bar{f}$ is well-defined.

It is straightforward to show that $\bar{f}$ is a K-algebra homomorphism. We will omit the laborious computations here, but the astute reader will note that they follow more or less immediately from the definition of $\bar{f}$.

We want to show that $\bar{f}^{\times} \circ \iota = f$. Letting $g_k \in C_n$ we have $\iota(g_k) = x^k + \langle x^n - 1 \rangle$. Then

$$\bar{f}^{\times}(x^k + \langle x^n - 1 \rangle) = (f(g_1))^k.$$

This is equivalent to $f((g_1)^k) = f(g_k)$. Hence, $\bar{f}^{\times} \circ \iota = f$, which was to be shown.

Finally, we must show that $\bar{f}$ is unique. Assume that there is some morphism $\bar{f}' : K[x]/\langle x^n - 1 \rangle \to S$ such that $\bar{f}'^{\times} \circ \iota = \bar{f}^{\times} \circ \iota = f$. Then

$$\bar{f}'^{\times}(x^k + \langle x^n - 1 \rangle) = f(g_k).$$

Hence, $\bar{f}$ and $\bar{f}'$ agree on powers of $x + \langle x^n - 1 \rangle$. Elements in $K[x]/\langle x^n - 1 \rangle$ may be represented as K-linear combinations of powers of $x + \langle x^n - 1 \rangle$:

$$p(x) + \langle x^n - 1 \rangle = a_0 + a_1 x + ... + a_{n-1}x^{n-1} + \langle x^n - 1 \rangle$$

for $a_0, a_1, ..., a_{n-1} \in K$. Because$\bar{f}'$ is a K-algebra homomorphism:

$$\begin{aligned} \bar{f}'(p(x) + \langle x^n - 1 \rangle) &= \bar{f}'(a_0 + a_1 x + ... + a_{n-1}x^{n-1} + \langle x^n - 1 \rangle) \\ &= a_0 + a_1 f(g_1) + ... + a_{n-1} f(g_{n-1}) \end{aligned}$$

which is equal to the image of $p(x)+\langle x^n - 1 \rangle$ under $\bar{f}$. Hence, $\bar{f}$ and $\bar{f}'$ agree on all K-linear combinations of powers of $x + \langle x^n - 1 \rangle$, which is the same as their domains, so $\bar{f} = \bar{f}'$.

Hence, $\bar{f}$ is the unique morphism making the diagram in Figure 5.1 commute.

$$\begin{tikzcd} C_n \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & K[x]/\langle x^n - 1 \rangle^{\times} \arrow[d, dotted, "\bar{f}^{\times}"] & K[x]/\langle x^n - 1 \rangle \arrow[d, dotted, "\bar{f}"] \\ & S^{\times} & S \end{tikzcd}$$

Figure 5.1: Diagram showing $\bar{f}$ and its restriction $\bar{f}^{\times}$.

Because $K[x]/\langle x^n - 1 \rangle$ satisfies the universal property of the group ring, it is canonically isomorphic to the group ring. $\square$

Proposition 5.2. *Let A, B, X, Y be objects in some category $\mathcal{C}$. Let $X \times Y$ and $A \times B$ be objects in $\mathcal{C}$ equipped with morphisms:*

$$\begin{aligned} \pi_1 &: X \times Y \to X \\ \pi_2 &: X \times Y \to Y \\ \sigma_1 &: A \times B \to A \\ \sigma_2 &: A \times B \to B \end{aligned}$$

and suppose that $X \times Y$ and $A \times B$ satisfy the universal property of products in $\mathcal{C}$. Additionally, let $\omega_1 : A \to X$ and $\omega_2 : B \to Y$ be isomorphisms. Then $X \times Y$ and $A \times B$ are necessarily isomorphic.

Proof. By the universal property of the product, there are unique morphisms $f : A \times B \to X \times Y$ and $\bar{f} : X \times Y \to A \times B$ such that

$$\begin{aligned} \pi_1 \circ f &= \omega_1 \circ \sigma_1 \\ \pi_2 \circ f &= \omega_2 \circ \sigma_2 \\ \sigma_1 \circ \bar{f} &= \omega_1^{-1} \circ \pi_1 \\ \sigma_2 \circ \bar{f} &= \omega_2^{-1} \circ \pi_2 \end{aligned}$$

$$
\begin{tikzcd}
B \arrow[rr, bend left, "\omega_2"] & & Y \arrow[ll, bend left, "\omega_2^{-1}"] \\
A \times B \arrow[u, "\sigma_2"] \arrow[d, "\sigma_1"'] \arrow[rr, dashed, bend left, "f"] & & X \times Y \arrow[u, "\pi_2"'] \arrow[d, "\pi_1"] \arrow[ll, dashed, bend left, "\bar{f}"] \\
A \arrow[rr, bend left, "\omega_1"] & & X \arrow[ll, bend left, "\omega_1^{-1}"]
\end{tikzcd}
$$

Figure 5.2: The products $A \times B$, $X \times Y$ and their projections.

In other words, the diagram in Figure 5.2 commutes in the forward and backward directions. Composing f and $\bar{f}$, we obtain morphisms $f \circ \bar{f} : X \times Y \to X \times Y$ and $\bar{f} \circ f : A \times B \to A \times B$.

By the commutativity of the diagram, and the fact that ω_1 and ω_2 are isomorphisms, we find that

$$
\begin{aligned}
\pi_1 &= \pi_1 \circ f \circ \bar{f} \\
\pi_2 &= \pi_2 \circ f \circ \bar{f} \\
\sigma_1 &= \sigma_1 \circ \bar{f} \circ f \\
\sigma_1 &= \sigma_1 \circ \bar{f} \circ f
\end{aligned}
$$

hence the diagram in Figure 5.3 commutes.

$$
\begin{tikzcd}
 & A \times B \arrow[dl, "\sigma_1"'] \arrow[d, "\bar{f}\circ f"] \arrow[dr, "\sigma_2"] & \\
A & A \times B \arrow[l, "\sigma_1"] \arrow[r, "\sigma_2"'] & B
\end{tikzcd}
$$

Figure 5.3: The composition $\bar{f} \circ f$ and projections from $A \times B$.

By the universal property of the product $A \times B$, the morphism $\bar{f} \circ f$ must be the unique such morphism. However, we find also that $\mathrm{id}_{A \times B}$ makes the same diagram commute. Hence $\bar{f} \circ f = \mathrm{id}_{A \times B}$.

We may apply a similar line of reasoning to $X \times Y$ and its projections, finding that $f \circ \bar{f} = \mathrm{id}_{X \times Y}$. Hence, f and $\bar{f}$ are inverse morphisms, and $X \times Y$ and $A \times B$ are isomorphic. □

6 Matrix representation

It is also possible to represent $\mathbb{T}$ as a matrix ring. In fact, there are multiple ways to do this. We describe one of them in this section.

As discussed earlier in the paper, we may view $\mathbb{T}$ as the group ring $\mathbb{R}[C_3]$. [1] details a process for constructing a matrix ring from a given group ring. Applied to $\mathbb{R}[C_3]$, this process leads us to the following representation.

We define the map $\Gamma : \mathbb{T} \to \mathbb{M}_3(\mathbb{R})$ by

$$
\Gamma : u + vi + wj \longmapsto \begin{bmatrix} u & v & w \\ w & u & v \\ v & w & u \end{bmatrix}
$$

where $u, v, w \in \mathbb{R}$.

The determinant of the matrix representation of a triplex number is related to the product of the number with its strong conjugate as follows:

$$
zz^* = \det \Gamma(z).
$$

This is a property shared with the matrix representation of complex numbers.

7 Future directions

In this paper, we have examined some of the properties of the triplex numbers, and we have gained especially rich insights about the structure of $\mathbb{T}$ by looking through the lens of $\mathbb{R} \times \mathbb{C}$. In particular, we have learned a great deal about triplex roots of polynomials and the nature of their relationship with real and complex roots.

Future work with the triplex numbers could take one of several directions. Our results about triplex roots of polynomial equations, while insightful, are also broad. As was noted in Section 1, it may be interesting to consider whether triplex numbers facilitate solving certain kinds of polynomial equations. It would seem natural to look at roots of cubic polynomials in $\mathbb{T}$, considering that the cubic roots of unity are in some sense directly encoded in the triplex numbers as basis elements.

The author is also inclined to attempt to develop some theory of analysis for triplex functions. In some preliminary work, the author was able to develop a notion of differentiation for triplex-valued functions and state triplex analogues of the Cauchy-Riemann equations, with which one can show, for example, that polynomial functions on $\mathbb{T}$ are differentiable everywhere, as with the complex numbers.

It also appears to be possible to define triplex analogues of the Wirtinger derivatives. However, the triplex Wirtinger operators fail to satisfy some important properties, the extent of which appears to be related to the way in which the notion of triplex conjugation diverges into the two distinct concepts of weak and strong conjugation. In the complex numbers, these concepts can be seen to coincide in the same action, which appears to explain some of the well-behavedness of the complex Wirtinger operators. Future work might aim to elucidate this relationship.

References

[1] Ted Hurley. Group rings and rings of matrices. *Int. J. Pure Appl. Math*, 31(3):319–335, 2006.